An Introduction to Type Theory

Felix Bradley

Department of Computer Science
Royal Holloway, University of London

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Presentation Overview

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A Brief History of Types

- Let's rewind back to 300 BC...
- Ancient Greek mathematician Euclid studied geometry, and published his work in *Elements* [Euclid, c. 300 BC].
- His work on geometry relied on five axioms; Euclid's *postulates*.
- Life was good...
- ...for approximately 2000 years.
- In 1868, Italian mathematician Eugenio Beltrami proved that Euclid's *parallel postulate* is independent of the other axioms.
- Is Euclidean geometry consistent?
- Is *anything* we've done so far correct?

A Brief History of Types

- German mathematician Gottlob Frege published his *Begriffschrift* in 1879
- Frege identified issues with prior work on logic and made a proposal:
- There's some basic rules we should expect of logic: consistency, completeness, and decidability.
  1) Consistency: you cannot prove contradictory propositions
  2) Completeness: every proposition is either provable or refutable
  3) Decidability: every proposition can be tested and every proof can be derived

A Brief History of Types

- The year is 1901.
- Along comes British mathematician Bertrand Russell.
- Russell asks a simple question:
  Does the set of all sets that do not contain themselves contain itself?

$$x := \{s | s \notin s\} \qquad x \in x?$$

- Modern mathematics says, "Trick question! That set isn't definable!"

"Unless it is, in which case: trick question! It's not a set, it's a *proper class*!"

- Is the question well-defined? Is it complete? Are we missing something?
- There's a rush to solve this problem; to find a complete, consistent and decidable basis for all mathematics.

A Brief History of Types

- It gets worse.
- Theorem: There exists mathematical objects which cannot be explicitly calculated or described.
- Theorem: There exists theorems in Peano arithmetic that cannot be proven with Peano arithmetic.
- It gets even worse.
- Along comes Austrian-Hungarian mathematician Kurt Gödel.
- Gödel publishes his *incompleteness theorems.*
  1) If a system of logic is consistent, then it is incomplete.
  2) A system of logic cannot prove it's own consistency.

A Brief History of Types

- So why bother?
- Russell demonstrated that naïve set theory is inconsistent; unrestricted set specification leads to contradiction.
- These questions led to the development of Zermelo-Frankael Set Theory (ZF) and the study of powerful tools such as the Axiom of Choice.
- Many other systems and bases of logic were studied and developed during this time.
- Russell's own proposal for solving this problem was a theory of *types*. (1908)
- Every mathematical object has a 'type', and these types form a hierarchy. Each type and its objects can be constructed only using subtypes.

Introducing Lambda Calculus

- In the 1930s, American mathematician Alonzo Church introduced lambda calculus.
- What is $f(x)$? What does $f(1)$ mean, if we only have $f(x)$ defined?
- $f(1)$ means to take all the xs in $f(x)$ and replace them with 1s.
- Church formalised by introducing λ and the notion of reduction

$$f(x) = x^2 + 2x + 3 \Rightarrow f := \lambda x.(x^2 + 2x + 3)$$

$$1 + 1 = 2$$

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$$1 + 1 \neq 2 \Rightarrow 1 + 1 \twoheadrightarrow^* 2$$

Introducing Lambda Calculus

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- As an abstract calculus, lambda calculus is a universal model for computation.
- It's a powerful enough system to represent arithmetic and logical operations.
- As a logic, lambda calculus was shown by Kleene and Rosser to be inconsistent.

Examples of Lambda Calculus

- Consider the function

$$\omega := \lambda x.(xx)$$

- Let's consider some examples:
- $\omega(2) \rightharpoonup (\lambda x.(xx))(2) \rightharpoonup (xx)[2/x] \rightharpoonup 2(2)$
- $\omega(\omega) \rightharpoonup (\lambda x.(xx))(\omega) \rightharpoonup (xx)[2/x] \rightharpoonup \omega(\omega)$

Simply Typed Lambda Calculus

- How did Church solve this problem? Type theory!
- Church's Simply Typed Lambda Calculus restricted lambda calculus to only those terms which could be *typed.*
- $2 : \mathbb{N} \qquad (+) : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$
- $d := \lambda(x : \mathbb{N}).2x$
- $d(2) \rightharpoonup^* 4$
- $d(\text{"}apple\text{"})$?
- $\omega := \lambda x.(xx)$ can't be typed.
- This restricted lambda calculus *is* logically consistent.

The Curry-Howard Correspondence

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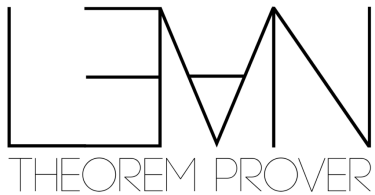
- By 1970, Mathematician Haskell Curry and logician William Alvin Howard realise something remarkable.
- Certain proof systems, such as natural deduction, behave kind of like an intuitionistic version of lambda calculus.
- This leads to the Curry-Howard Correspondence: that the act of writing a mathematical proof *has a one-to-one correspondence* with the act of writing a computer program.
- If A and B are propositions, then I may write a proof p that $A \to B$.
- If A and B are types, then I may describe a function $f : A \to B$ that takes in an object of type A and returns an object of type B.
- If we interpret *propositions as types*, then writing a proof is the same as describing a function.

Proof Assistants

- The Curry-Howard Correspondence has lead to the development of *proof assistants*.
- Proof assistants are advanced programming languages.
- You can formalise mathematics and write up your proofs.
- The proof assistant can verify your proof and ensure that you are correct.
- Mathematicians like Kevin Buzzard and Terence Tao are pushing proof assistants with cutting-edge research.
  [Buzzard, 2024, Tao, 2024].



Programming Languages

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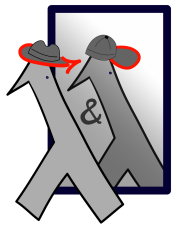
- In fact, type theory has worked its way into every popular programming language.
- It stops you, the programmer, from making mistakes.
- Type-driven programming paradigms help to prevent bugs.

In Conclusion

- Type theory was developed as an alternative to set theory.
- Type theory has bloomed into something new and exciting.
- Type theory is an active area of research.
- We're trying to get students more involved with type theory!



Proof and Programs Club @ RHUL

Thank you for your time!

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